Derivatives

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}e^x = e^x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}a^x = e^x \ln a$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan^{-1} x = \frac{1}{(x^2+1)}$	$\frac{d}{dx}\ln x = \frac{1}{x}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$	$(fg)' = f'g + g'f$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\cot^{-1} x = \frac{-1}{(x^2+1)}$	$(\frac{f}{g})' = \frac{f'g-g'f}{g^2}$

Integrals (remember +C)

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$	$\int \tan x\,dx = -\ln\lvert\cos x\rvert$	$\int \tan x \sec x\,dx = \sec x$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert$	$\int \cot x\,dx = \ln\lvert\sin x\rvert$	$\int \cot x \csc x\,dx = -\csc x$
$\int e^x\,dx = e^x$	$\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert$	$\int x \sin x\,dx = \sin x - x\cos x$
$\int a^x\,dx = \frac{1}{\ln a}a^x$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert$	$\int x\cos x\,dx = \cos x + x\sin x$
	$\int \ln x\,dx = x\ln x - x \ (x > 0)$	

For $a^2 \neq b^2$:

$$\int \sin ax \sin bx\,dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)}$$

$$\int \cos ax \cos bx\,dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)}$$

$$\int \sin ax \cos bx\,dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}$$

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

or: $\int fg = f\int g - \int(f'\int g)$

Vector Calculus

Gradient, ∇f:

$\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$.

$\nabla f(x)$ points in the direction of increasing f.

Directional Derivative, $D_{\mathbf{v}}f$:

∇f at x in the direction of $\mathbf{v}$:

$D_{\mathbf{v}}f(x) = \nabla f(x) \bullet \frac{\mathbf{v}}{||\mathbf{v}||}$

Curl, curl$\mathbf{F} = \nabla \times \mathbf{F}$:

for $\mathbf{F} = (F_1, F_2, F_3)$,

$\nabla \times \mathbf{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$

Divergence, div$\mathbf{F} = \nabla \bullet \mathbf{F}$:

for $\mathbf{F} = (F_1, F_2, F_3)$,

$\nabla \bullet \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Identities & Properties:

$\nabla(fg) = f\nabla g + g\nabla f$	$\mathrm{div}(\nabla f \times \nabla g) = 0$
$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \bullet \nabla g)$	$\mathrm{curl}(\nabla f) = 0$
$\nabla(\frac{f}{g}) = \frac{g\nabla f - f\nabla g}{g^2}$	$\mathrm{div}(\mathrm{curl}\mathbf{F}) = 0$
$\mathrm{curl}(\mathbf{F} + \mathbf{G}) = \mathrm{curl}\mathbf{F} + \mathrm{curl}\mathbf{G}$	$\mathrm{div}(\mathbf{F} + \mathbf{G}) = \mathrm{div}\mathbf{F} + \mathrm{div}\mathbf{G}$
$\mathrm{curl}(f\mathbf{F}) = f\mathrm{curl}\mathbf{F} + \nabla f \bullet \mathbf{F}$	$\mathrm{div}(f\mathbf{F}) = f\mathrm{div}\mathbf{F} + \nabla f \bullet \mathbf{F}$
$\mathrm{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \bullet \mathrm{curl}\mathbf{F} - \mathbf{F} \bullet \mathrm{curl}\mathbf{G}$	$\mathrm{div}(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

Green's Theorem:

$\oint_C P\,dx + Q\,dy = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,dA$

Divergence Theorem:

$\oiint_S \mathbf{F} \bullet \hat{\mathbf{n}}\,dS = \iiint_V \mathrm{div}\mathbf{F}\,dV$

Useful Limits

$\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$ $\quad \lim_{n\to 0}(1+n)^{\frac{1}{n}} = e$ $\quad \lim_{n\to\infty}(n)^{\frac{1}{n}} = 0$ $\quad \lim_{n\to\infty}(x)^{\frac{1}{n}} = 1,\ x > 0$

$\lim_{x\to 0}\frac{-1+\cos x}{x} = 0$ $\quad \lim_{x\to 0}\frac{\sin x}{x} = 1$ $\quad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

L'Hôpital's Rule: if $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ if $g'(x) \neq 0$.

Useful Sums

$\sum_{i=1}^n c = nc$ $\quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$ $\quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum_{i=1}^n i^3 = (\frac{n(n+1)}{2})^2$

Basic Trigonometry & The Unit Circle

$\sin\theta = y/r \qquad \cos\theta = x/r \qquad \tan\theta = \sin\theta/\cos\theta$

$\csc\theta = 1/\sin\theta \qquad \sec\theta = 1/\cos\theta \qquad \cot\theta = 1/\tan\theta$

Values of trigonometric functions for common angles:

θ (rads)	$0, 2\pi$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Trigonometric Identities & Relationships

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\tan^2\theta = \sec^2\theta - 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cot^2\theta = \csc^2\theta - 1$

$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

Sine Law: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ Cosine Law: $c^2 = a^2 + b^2 - 2ab\cos C$

Complex Numbers

Imaginary Unit: $i = \pm\sqrt{-1}$

De Moivre's Formula: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Linear Algebra

Basic Determinants:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} r & s & t \\ u & v & w \\ x & y & z \end{vmatrix} = \begin{matrix} r(vz-wy) \\ -s(uz-wx) \\ +t(uy-vx) \end{matrix}$$

Eigenvalues & Eigenvectors:

Eigenvalues λ of the matrix A are found by solving the *characteristic equation*: $\det(A-\lambda I) = 0$ for λ.
Eigenvectors $\mathbf{v}$ are found by solving $(A-\lambda I)\mathbf{v} = 0$ for $\mathbf{v}$, for each value of λ.
Check: Vector $\mathbf{v}$ is an eigenvector of matrix A, with eigenvalue λ, if $A\mathbf{v} = \lambda\mathbf{v}$.

Cramer's Rule:
for $A\mathbf{x} = \mathbf{b}$, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$,
$x_i = \frac{\det(A_i)}{\det(A)}$, where $A_i = A$ with its i^{th} column replaced with $\mathbf{b}$.

$$\text{i.e. } A_i = \left[\begin{array}{ccc|c|ccc} a_{1,1} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,i-1} & b_m & a_{m,i+1} & \cdots & a_{m,n} \end{array}\right]$$

Geometry in n-Space (for $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \in \mathbb{R}^n$)

Length (Euclidian norm), $||\mathbf{a}||$:

$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$||k\mathbf{a}|| = |k|||\mathbf{a}||$

Dot Product, $\mathbf{a}\bullet\mathbf{b}$:

$\mathbf{a}\bullet\mathbf{b} = \mathbf{b}\bullet\mathbf{a} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\mathbf{a}\bullet(\mathbf{b}+\mathbf{c}) = \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{c} \qquad ||\mathbf{a}|| = \sqrt{\mathbf{a}\bullet\mathbf{a}}$

NOTE: $\mathbf{a}$,$\mathbf{b}$ are *orthogonal* if $\mathbf{a}\bullet\mathbf{b} = 0$

Angle θ between $\mathbf{a}$ and $\mathbf{b}$:

$\cos\theta = \frac{\mathbf{a}\bullet\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}$

Cross Product, $\mathbf{a}\times\mathbf{b}$:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix} \qquad ||\mathbf{a}\times\mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}||\sin\theta$$

Projection of $\mathbf{b}$ onto $\mathbf{a}$, $\text{proj}_{\mathbf{a}}\mathbf{b}$:

$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{(\mathbf{b}\bullet\mathbf{a})}{||\mathbf{a}||^2}\mathbf{a}$

Projection of $\mathbf{b} \perp \mathbf{a}$, $\text{perp}_{\mathbf{a}}\mathbf{b}$:

$\text{perp}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$

Logarithms and Exponents

$\log_b(xy) = \log_b(x) + \log_b(y)$ $\quad$ $\log_b(x^y) = y\log_b(x)$ $\quad$ $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$

$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$ $\quad$ $\log_b(\sqrt[y]{x}) = \frac{\log_b(x)}{y}$ $\quad$ $\log_a b = \frac{\log_c b}{\log_c a} = \frac{\ln b}{\ln a}$ (change the base)

Power & Taylor Series

Taylor Series approximation of f(x):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$
$$= \left[\sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!}(x-c)^n\right] + R_k(c,x)$$

Taylor Remainder, $R_k(c,x)$:

$$R_k(c,x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}, c \lesseqgtr \xi \lesseqgtr x$$

NOTE: $\lim_{k\to\infty} R_k(c,x) = 0$ if series converges.

Some Useful Power Series:

➔ FOR ALL x:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}x^{2n}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

➔ FOR $|x| < 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Radius of Convergence (RoC):

RoC R for a power series $\sum_{n=0}^{\infty} a_n x^n$: $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$

Fourier Series (on the interval $[-p,p]$)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(\frac{n\pi x}{p}) + b_n\sin(\frac{n\pi x}{p})\right]$$

$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$ $\quad$ $a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos(\frac{n\pi x}{p})\,dx$ $\quad$ $b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin(\frac{n\pi x}{p})\,dx$

Laplace Transforms ($F(s) = \mathscr{L}\{f(t)\}$)

$f(t)$	$\Leftrightarrow F(s)$	$f(t)$	$\Leftrightarrow F(s)$	$f(t)$	$\Leftrightarrow F(s)$
$f(t)$	$\int_0^{\infty} e^{-st}f(t)\,dt$	$\delta(t)$	1	1	$\frac{1}{s}$
(periodic, T) $f(t)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$	t^n	$\frac{n}{s^{n+1}}$	e^{-at}	$\frac{1}{s+a}$
$u(t-T)f(t-T)$	$e^{-sT}F(s)$	$g(f(t))$	$F(s)G(s)$	$\sin(at)$	$\frac{a}{s^2+a^2}$
$f'(t)$	$sF(s) - f(0^+)$	$e^{at}f(t)$	$F(s-a)$	$\cos(at)$	$\frac{s}{s^2+a^2}$

Engineering Economics

P=Present Value F=Future Value A=Annuity G=Gradient i=Interest Rate(%) N=# of Interest Periods

$\frac{F}{P} = (1+i)^N$ $\quad$ $\frac{A}{P} = \frac{i(1+i)^N}{(1+i)^N-1}$ $\quad$ $\frac{A}{F} = \frac{i}{(1+i)^N-1}$ $\quad$ $\frac{A}{G} = \frac{1}{i} - \frac{N}{(1+i)^N-1}$

$\text{NPV} = \sum_{t=0}^{N} \frac{C_t}{(1+i)^t}$ $\quad$ $\text{RORC} = (1+i)(\frac{\text{NPV}}{\text{PWCE}}+1)^{\frac{1}{N}} - 1$

Numerical Methods

Absolute Error: $e_{\text{abs}} = |x - \overline{x}|$

Relative Error: $e_{\text{rel}} = |\frac{x-\overline{x}}{x}|$

Newton's Method: (root-finding)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Euler's Method (Solving ODEs):

given $f(x,y) = \frac{dy}{dx}$, x_0, y_0, h = step size:
$y_{k+1} = y_k + hf(x_k, y_k)$, where $x_k = kh + x_0$.

Geometric Shapes & Solids

A =Area, C = Circumference, A_S = Surface Area, V = Volume

$A = \frac{1}{2}bh$	$A = \pi r^2$	$A_S = 4\pi r^2$	$A_S = 2\pi r(h+r)$	$A_S = \pi r(r + \sqrt{h^2+r^2})$
	$C = 2\pi r$	$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$